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LETTER TO THE EDITOR

Continuum model for growth with number conservation: the dynamic critical exponent in the spherical limit

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Abstract. The continuum model describing growth with conservation of particle number requires an infinite number of nonlinear terms for a one-dimensional substrate. By considering an N-vector generalization of this growth model, we show that the dynamic critical exponent $z = 4$ in the $N \to \infty$ limit. This spherical limit provides the only universal exponents for one-dimensional growth models with conserved particle number but non-conserved noise.

Deposition problems have been widely studied [1–10] in the last few years. The model which we want to discuss here is that for an ideal molecular beam epitaxy (MBE) process. Atoms from a hot source impinge perpendicularly upon the surface of a perfect substrate (D-dimensional) with the growth of the resulting thin film occurring in a lattice-gas-solid-on-solid fashion without any desorption and bulk vacancy/surface overhang formation. Ideal MBE growth is a competition between the kinetic process of random deposition and surface diffusion. The shot noise inherent in the random deposition process produces kinetic roughness at the growth front whereas surface diffusion tends to smooth the growing interface. Models to portray this competition were introduced independently by Wolf and Villain [3] and Das Sarma and Tamborenea (DT) [4]. A continuum language description involves finding the evolution equation for the height variable $h(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ is a D-dimensional vector giving the location on the substrate. The evolution equation takes the form

$$\frac{\partial h}{\partial t} = F(h, \boldsymbol{x}, t). \tag{1}$$

Surface diffusion implies that the deposited particles generate a macroscopic current $\boldsymbol{j}(x, t)$ which is a vector parallel to the average surface direction. Local changes in the surface height are caused by the non-zero surface currents and, since the total number of particles remains unchanged during the diffusion process, the current obeys a continuity equation making $\boldsymbol{F} = -\boldsymbol{\nabla} \cdot \boldsymbol{j}$. With the surface current driven by the local chemical potential, $\boldsymbol{j} = -\boldsymbol{\nabla}\mu$. To find the chemical potential we note that μ cannot depend on h, but is determined by the number of bonds that need to be broken for surface diffusion. The number of bonds is determined by the number of nearest neighbours and hence by the local curvature, i.e. a negative curvature means fewer neighbours and a greater surface current. Hence $\mu \propto -\nabla^2 h$ and equation (1), on inclusion of the randomness in the incoming flux, becomes

$$\frac{\partial h}{\partial t} = -K\nabla^4 h + \eta \tag{2}$$

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(it is assumed that processes contributing $\nabla^2 h$ to the equation of motion are suppressed) where η is a random field specified by the correlation

$$\langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t')\rangle = 2D_0\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'). \tag{3}$$

The above linear equation (equation (2)) is easily analysed. To capture the essence of the discrete DT model, however, nonlinear terms have to be included and the vital point to note as pointed out by Krug [11] is that the DT model does not have tilt invariance. Thus the nonlinear contribution to the chemical potential can be written as a power series in $(\boldsymbol{\nabla} h)$ and we have, for the continuum version [12, 13] of the DT model,

$$\frac{\partial h}{\partial t} = -K\nabla^4 h - \nabla^2 \sum_{n=1} \lambda_n (\boldsymbol{\nabla} h)^{2n} + \eta. \tag{4}$$

Characterization of the form of the surface can be achieved by means of the two-point correlation function,

$$C(k, \omega) = \frac{\langle h(k, \omega)h(k', \omega)\rangle}{(2\pi)^{D+1}\delta^D(\boldsymbol{k} + \boldsymbol{k})\delta(\omega + \omega')} \tag{5}$$

and the response function

$$G(k, \omega) = \frac{1}{\delta^D(\boldsymbol{k} + \boldsymbol{k}')\delta(\omega + \omega')} \left\langle \frac{\partial h(k, \omega)}{\partial \eta(k', \omega)} \right\rangle. \tag{6}$$

The scaling properties of the model are expressed in the form of the scaling laws

$$\begin{aligned} C(k, \omega) &= k^{-(2\alpha + D + z)} f\left(\frac{\omega}{k^z}\right) \\ G(k, \omega) &= k^{-z} g\left(\frac{\omega}{k^z}\right). \end{aligned} \tag{7}$$

The two exponents α and z describe the spatial and temporal scalings in the problem. The exponents of the model shown in equation (4) were calculated by Lai and Das Sarma [5], who considered all the terms except the first (λ_1) to be irrelevant and the exponents were found to be

$$z = \tfrac{1}{3}(8 + D) \qquad \alpha = \tfrac{1}{3}(4 - D). \tag{8}$$

At $D = 1$, equation (8) yields $z = 3$ and $\alpha = 1$. The numerical simulation of the DT model [14], however, showed that in $D = 1$, $z = 4$ and $\alpha \simeq 1.5$ in striking contrast to those inferred from equation (8). At $D = 2$, on the other hand, the simulation results in the DT model indeed agree with the results of equation (8). The solution to this puzzle was speculated upon by Krug [11] who argued that in $D = 1$ all the nonlinear terms in equation (4) will be relevant while in $D = 2$ only the λ_1 term is relevant. This point of view received further support when it was shown [15] that in $D = 1$ there is an infrared divergence and the divergence gets stronger with increasing n as we consider the different nonlinear terms in equation (4). A summation of the leading logs would change the exponents in a non-universal manner but explicit values for α and z could not be obtained. In this work, we generalize equation (4) for the scalar h to N-dimensional vectors $\boldsymbol{h}$ and show that as $N \to \infty$, $z \to 4$ exactly. This limit is universal.

The generalization of equation (4) will be carried out following the random coupling approach of Kraichnan [16]. This is a N-vector generalization which is, as shown by Eyink [17], equivalent to the approaches of Mou and Weichman [18] and Doherty *et al* [19] in the context of turbulence and KPZ-like equations, respectively. To implement Kraichnan's

scheme, we introduce N identical systems each with their own variable h and then introduce the collective coordinates

$$h_\alpha = \frac{1}{N^{1/2}} \sum_n \mathrm{e}^{2\pi \mathrm{i}\alpha/n/N} h_{[n]} \tag{9}$$

where $n = 1, 2, \ldots, N$ and $\alpha = 0, \pm 1, \pm 2, \ldots, \pm s$, with $2s + 1 = N$.

In terms of h, equation (4) becomes

$$\frac{\partial h_\alpha}{\partial t} = -K\nabla^4 h_\alpha - \nabla^2 \sum_n \sum_{\beta_1,\beta_2,\ldots,\beta_{n-1}} \frac{\lambda_n}{N^{n+1/2}} (\boldsymbol{\nabla} h)_{\beta_1} (\boldsymbol{\nabla} h)_{\beta_2} (\boldsymbol{\nabla} h)_{\beta_3} (\boldsymbol{\nabla} h)_{\beta_4} \cdots (\boldsymbol{\nabla} h)_{\alpha - \sum_i \beta_i} + \eta_\alpha.$$

One now introduces the random coupling through a phase factor which makes the above equation

$$\frac{\partial h_\alpha}{\partial t} = -K\nabla^4 h_\alpha - \nabla^2 \sum_n \sum_{\beta_1,\beta_i,\ldots,\beta_{n-1}} \mathrm{e}^{\mathrm{i}\theta_{\beta_1,\beta_2,\ldots,\beta_{n-1},\alpha-\sum_i \beta_i}} \frac{\lambda_n}{N^{n+1/2}} (\boldsymbol{\nabla} h)_{\beta_1} (\boldsymbol{\nabla} h)_{\beta_2} \cdots (\boldsymbol{\nabla} h)_{\alpha - \sum_i \beta_i} + \eta_\alpha \tag{10}$$

with $\langle \eta_\alpha \eta_\beta \rangle = 2D_0 \delta_{\alpha\beta} \delta^D(\boldsymbol{r} - \boldsymbol{r}') \delta(t - t')$ and the phase factor $\theta_{\beta_1,\beta_2,\ldots,\beta_{n-1},\alpha-\sum_i \beta_i}$ acquiring a random value between 0 and 2π for every assignment $\beta_1, \beta_2, \ldots, \beta_{n-1}$.

In the limit of $N \to \infty$, it is easy to check that diagrams such as those shown in figure 1 vanish and the self-energy and correlation functions are given by the series shown in figure 2, where, in the scaling limit, we have dropped the term $D_0|G(k,\omega)|^2$ in $C(k,\omega)$ and we will approximate the Greens function by $G^{-1}(k,\omega) = -\mathrm{i}\omega + \Sigma(k,\omega)$. Using Lorentzian approximations for $C(k,\omega)$ and $G(k,\omega)$, and working to one-loop accuracy (i.e. retaining only the one-loop diagrams of figure 2) we have, from the diagram for $\Sigma(k,\omega)$,

$$1 = 2\frac{\lambda_1^2 C}{\Gamma^3} \int_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \frac{\mathrm{d}^D p}{q^{2\alpha+D}} \cdot \frac{p^2 q^2 (\boldsymbol{p}\cdot\boldsymbol{k})}{p^z + q^z} \tag{11a}$$

and from the diagram for $C(k,\omega)$

$$1 = \frac{\lambda_1^2 C}{\Gamma^3} \int_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \frac{\mathrm{d}^D p}{p^{2\alpha+D} q^{2\alpha+D}} \frac{p^2 q^2}{p^z + q^z}. \tag{11b}$$

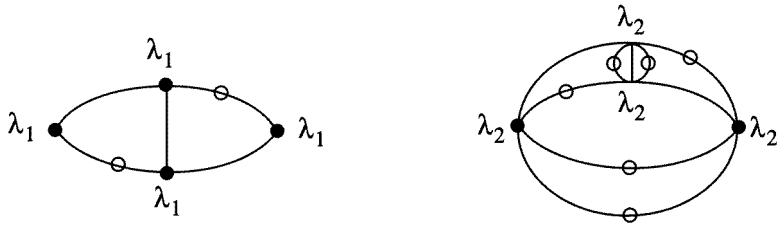

Figure 1. Examples of diagrams which would yield the same logarithmic divergence as the lowest order graphs in λ_1 and λ_2, but vanish in the $N \to \infty$ limit. The line with a circle is a correlator.

In writing down equations (11*a*) and (11*b*), the so-called mode coupling equations, we have simply followed Bouchaud and Cates [20] (the finer points of non-Lorentzian lineshapes and whether the zero frequency or the equal time part of $C(k,\omega)$ is more appropriate will not be considered here since they are irrelevant for the point we wish

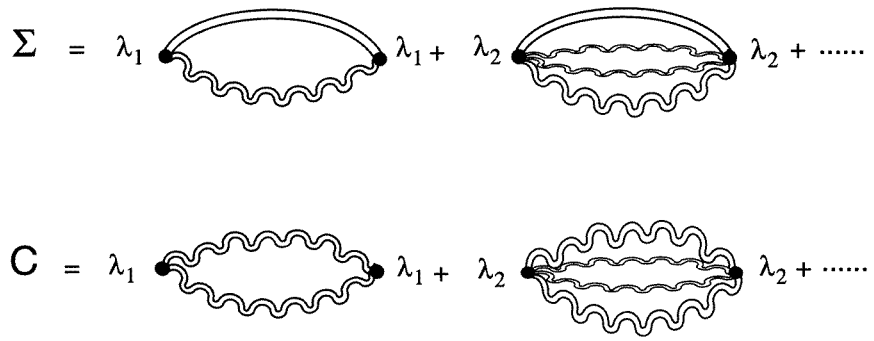

Figure 2. The self-consistent equation for the self-energy $\Sigma(k,\omega)$ and the correlation function $C(k,\omega)$ in the $N \to \infty$ limit. The double straight line is the dressed propagator and the double wavy line is the dressed correlator.

to make). In equations (11*a*) and (11*b*), all momenta are actually scaled by k and hence we can define the two numbers

$$I_1 = 2\int_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{1}} \frac{\mathrm{d}^D p}{q^{2\alpha+D}} \frac{p^2 q^2 (p\cdot k)}{p^z + q^z} \tag{12a}$$

and

$$J_1 = \int_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{1}} \frac{\mathrm{d}^D p}{p^{2\alpha+D} q^{2\alpha+D}} \frac{p^2 q^2}{p^z + q^z}. \tag{12b}$$

From equations (11*a*) and (11*b*), we find $I_1 = J_1$, which yields a relation between α and z. The overall scaling of the integrals gives $\alpha + z = 4$ and, thus, one can determine α and z. This method will work provided I_1 and J_1 are finite numbers. Inspection of I_1 and J_1 shows that this is true for $\alpha < 1$. For $\alpha \geqslant 1$, there is a low momentum singularity and the integrals diverge. The model reaches $\alpha = 1$ at $D = 1$ according to equation (8) and hence the mode coupling result of equating I_1 and J_1 is going to run into difficulty at $D = 1$. To study the behaviour as $\alpha \to 1$ our strategy will be to expand [21]

$$I_1 = \frac{C_1}{1-\alpha} + C_2 + C_3(1-\alpha) + \cdots \tag{13a}$$

and

$$J_1 = \frac{D_1}{1-\alpha} + D_2 + D_3(1-\alpha) + \cdots. \tag{13b}$$

Clearly, $C_1 = D_1 = 2$ and

$$C_2 = 2\int_{-\infty}^{\infty} \frac{\mathrm{d}q}{q} \frac{p^2(\boldsymbol{1}-\boldsymbol{p})}{p^z + |1-p|^z} - 2\int_{-1}^{1} \frac{\mathrm{d}q}{q} \tag{14}$$

while

$$D_2 = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{p|\boldsymbol{1}-\boldsymbol{p}|} \frac{1}{p^z + |1-p|^z} - \int_{-1}^{1} \frac{\mathrm{d}p}{p} - \int_{-1}^{1} \frac{\mathrm{d}q}{q}. \tag{15}$$

If we are going to get results consistent with equation (8) then, for $z = 3$, we must get $C_2 = D_2$. Explicit evaluation of the integrals show that $C_2 = 8\pi/9\sqrt{3}$ and $D_2 = 4\sqrt{3}\pi/3 - (16/3)\ln 2 \neq C_2$. This shows that the continuum model of Lai and Das Sarma [5] ($\lambda_1 \neq 0$ and all other λ's zero) will not yield $z = 3$ for $D = 1$.

To find the value of z in the $N \to \infty$ limit, we note that as $D \to 1(\alpha \to 1)$ all the diagrams are important. Focusing on Σ, the second diagram in figure 2 has a leading divergence of $(1-\alpha)^{-3}$ as $\alpha \to 1$ and hence is more important than the first diagram which goes as $(1-\alpha)^{-1}$. In fact the diagram corresponding to λ_n, i.e. the one with $2n$ internal lines, yields a leading divergence which is $(1-\alpha)^{-(2n-1)}$ and hence the higher the value of n, the stronger the contribution. One subtraction which removes the leading singularity yields a term proportional to $(1-\alpha)^{-(2n-2)}$, with the coefficient given by $(n/2)\cdot C_2$, i.e. for the diagram with $2n$ integral lines

$$I_n = \frac{n}{(1-\alpha)^{2n-1}} + \frac{nC_2/2}{(1-\alpha)^{2n-2}} + \cdots . \tag{16}$$

Considering the correlation function at this level,

$$J_n = \frac{n}{(1-\alpha)^{2n-1}} + \frac{D_2^{(n)}}{(1-\alpha)^{2n-2}} + \cdots \tag{17}$$

where $D_2^{(n)}$ is weaker than $\mathrm{O}(n)$. We must have $I_n = J_n$ as $n \to \infty$ for consistency and, since $D_2^{(n)}$ is not of $\mathrm{O}(n)$, equation (16) shows that $\mathrm{Lt}_{n\to\infty} nC_2$ has got to be finite and hence z has to be such that

$$C_2(z) = 0 \tag{18}$$

which yields $z = 4$.

Thus, in the limit $N \to \infty$ and $D \to 1$ (the double limit is vital), the mode coupling equations for $\Sigma(k,\omega)$ and $C(k,\omega)$ as expressed by the diagrammatic series in figure 2 are exact and exploitation of the infrared divergence of the integrals, as $D \to 1$, yields from self-consistency the value of the dynamic scaling exponent. Note that $N \to \infty$ gives the series and $D \to 1$ implies that the integral with $n \to \infty$ will dominate. This dominance assures the universality of z (the other exponent follows from hyperscaling to be $\alpha = 1.5$). The correction to the spherical limit induces corrections which depend upon the various coupling constants. This brings in non-universal features which have been captured by Dasgupta *et al* [22] in their numerical integration of systems similar to that given by equation (4). We need to emphasize that the connection between the spherical limit considered here and the $N = 1$ system is perhaps remote. However, the spherical limit helps establish universal answers in a situation where universality seems to be hard to come by.

A brief comment on dimensionalities different from one is in order. For $D > 1$, the integrals such as those in equations (11*a*) and (11*b*) are finite and the self-consistent power counting is appropriate. As can easily be checked from either of the above equations, this consistency requires $\alpha + z = 4$ and, taken together with the hyperscaling relation, one gets both the exponents of equation (8). The terms with λ_2, λ_3 etc are indeed irrelevant as can be easily checked by writing the analogue of equation (11*a*) for the λ_2 coupling alone. Self-consistent power counting together with hyperscaling now yields $z = 3 + \frac{3}{5}(D-1)$, which is higher than $z = 3 + \frac{1}{3}(D-1)$ of equation (8) and thus asymptotically the effect of λ_2 can be ignored. For $D > 1$, the exponents of Lai–Das Sarma hold. On the other hand the region $D < 1$ is certainly not approachable by the present method and indeed the generalization of the DT model [2] to Cantor sets would be interesting.

A final comment concerns the observation of anomalous scaling by Plischke *et al* [23], Schroeder *et al* [24] and Das Sarma *et al* [14], where the system size enters the scaling behaviour on $D = 1$. While our calculation of the exponent z has not addressed this issue, it should be mentioned that our pinpointing of the reason of failure of the standard analysis immediately clarifies the reason for this anomaly. As noted following equations (11*a*) and

(11*b*), the integrals have an infrared divergence in $D = 1$ and hence they would require a cut-off. This cut-off, which corresponds to a low momentum cut-off, is going to be ξ^{-1} if the correlation length $\xi(\sim t^{1/z})$ is small compared to the system size L and L^{-1} if $L \ll \xi$. Thus finite size effects creep in through the infrared divergence in $D = 1$.

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